Assessing algebraic form automatically

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Outline

- Why did I build STACK?
- What have I learned about algebraic form?
- STACK and algebraic form
- STACK and AI (because everyone is asking me!)

Why did I build STACK?

Assessment is the cornerstone of effective education.

- We need assessment worth teaching to.
- I believe universities (we) need to take responsibility for our important tools/software.

System demo

Demonstration of the software.

Materials designed for the format

Questions, quizzes and courses must be designed with the format in mind.

In this case, the “reveal” follow-on opens a template.

If false the give reason(s) below.

This does not test if students know potential reasons!

Encounters with proof

Valuable activities associated with proof.

Practical Online Assessment of Mathematical Proof:
http://arxiv.org/abs/2006.01581

Writing sequences of problems

... is something of an art form.
It is much easier to ask students to “prove this...”!

Adventures in algebra

Crowder, N. A. and Martin, G. C. (1960) *Adventures in Algebra*, Doubleday.

Students follow a non-linear path through the book.

Crowder

[from page 112]

YOUR ANSWER: Yes, Q_L is divisible by some prime number.

You are correct. In fact, Q_L, being the product of all the prime numbers, is divisible by *any* prime number, since it has all the prime numbers as factors.

Let's see what else we know about Q_L.

Is it an odd number or an even number?

Odd. **page 103**

Even. **page 108**

I don't know. **page 115**

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Crowder

YOUR ANSWER: I don't know whether Q_L is odd or even.

It's simple.

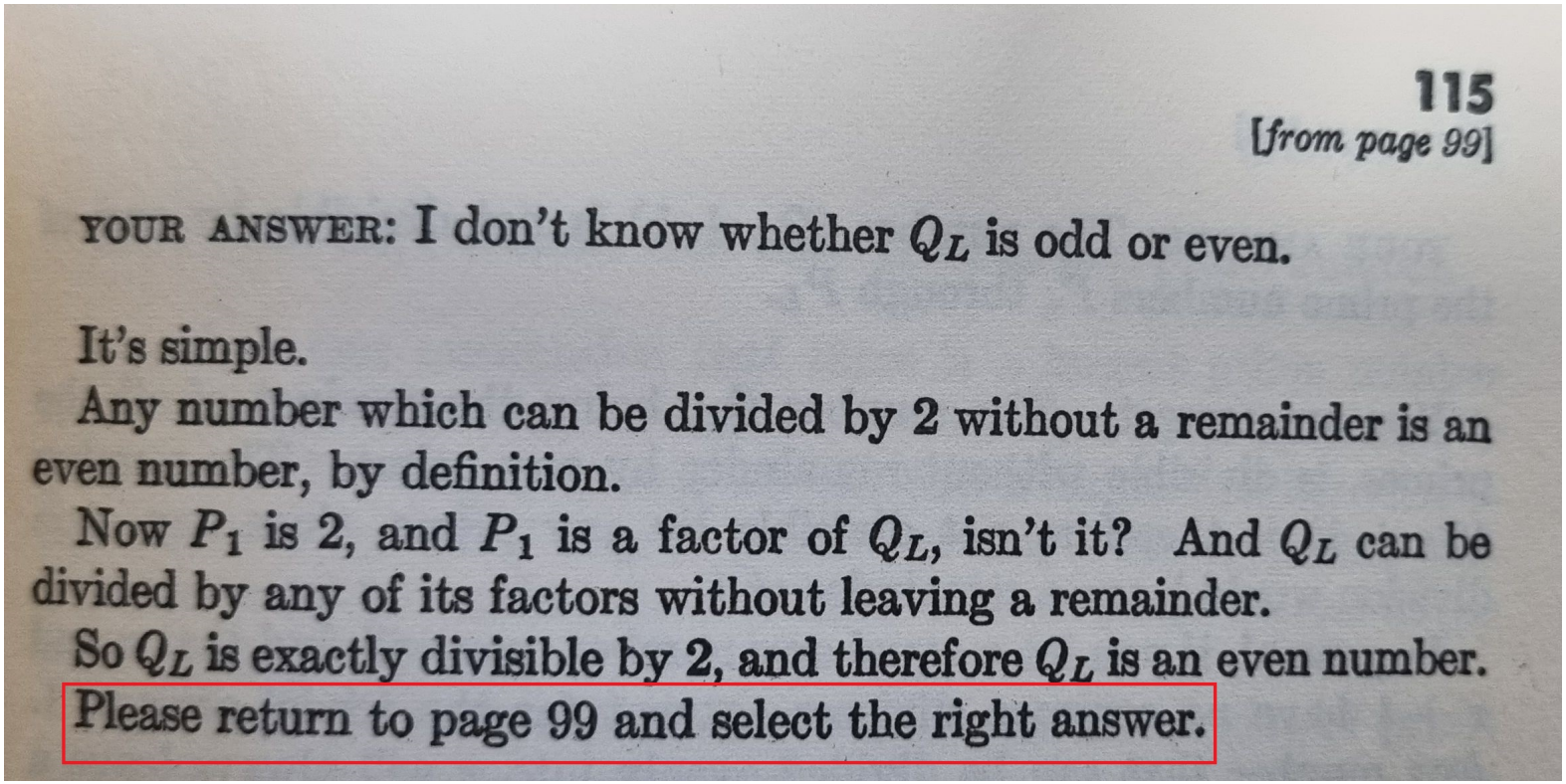
Any number which can be divided by 2 without a remainder is an even number, by definition.

Now P_1 is 2, and P_1 is a factor of Q_L, isn't it? And Q_L can be divided by any of its factors without leaving a remainder.

So Q_L is exactly divisible by 2, and therefore Q_L is an even number.

Please return to page 99 and select the right answer.

Crowder



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New in STACK 4.4.3

- The new `[[reveal]]` block.
- Descriptions in the question, PRTs, and test cases.
- Updated “STACK question dashboard”
- Question authors can write their own validation functions.

Establishing properties

How do we test if

1. a point is in a subspace?
2. two subspaces are “the same”?

Properties vs calculations

Classical CAS: `factor`

$$
\begin{aligned}
& x^8 + 16x^4 + 48 \\
= \; & (x^4 + 4)(x^4 + 12) \\
= \; & (x^2 + 2x + 2)(x^2 - 2x + 2)(x^4 + 12) \\
= \; & (x^2 + 2x + 2)(x^2 - 2x + 2)(x^2 + 2\sqrt[4]{3}x + 2\sqrt[4]{3})(x^2 - 2\sqrt[4]{3}x + 2\sqrt[4]{3} \\
= \; & (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)(x^4 + 12)
\end{aligned}
$$

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= \; & (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)(x^4 + 12)
\end{aligned}
$$

Someone made a subjective value judgement

Clearly...

"Clearly"

- $x^8 + 16x^4 + 48$ is a quadratic in x^4
- $cos^2(x) - 1$ is the difference of two squares
- We can factor 0 as $(x - |x|)(x + |x|)$

Noticing, and pattern recognition, is fundamental to mathematical work.

There is a continuum between recognising simple forms through to the outright obscure.

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

Pattern matching in general

$$ax^2 + bx + c \quad \equiv \quad 3z^2 - 1$$

$$\frac{1}{4b^2c^2}(4b^2c^2 - (b^2 + c^2 - a^2)^2)$$

as the difference of two squares

$$x^2 - y^2 = (x + y)(x - y)$$

Vocabulary....

We lack an agreed vocabulary for detailed discussion.

What is this rule called?

$$0 \times x \to 0$$

Equivalence in general

How do you recognise a constant of integration?

$$\int x^3 \, \mathrm{d}x = \frac{x^4}{4} + c$$

Show that the relations in each group are equivalent where the C's and A's denote arbitrary constants.

1.	$x + \log(y) = C,$	$ye^x = A.$
2.	$y + \log(3x) = C,$	$y + \log(x) = A.$
3.	$Cy = e^{x+2},$	$y = Ae^x.$
4.	$C + \sin^{-1}(x),$	$y = A - \cos^{-1}(x).$
5.	$y = C_1 \sin(x + C_2),$	$y = A_1 \sin(x) + A_2 \cos(x).$
6.	$\cosh(y) + \sinh(y) = Cx,$	$Ae^y = x.$

(Porter 1970)

In-built tools in STACK

STACK provides many function

- Checks an expression is in “factored form”
- Checks an expression is in “partial fraction form”
- Has a constant of integration

But some design choices have been made...

Efficient material production

Each question a carefully hand-crafted digital object

$$\leftrightarrow$$

Well-tested components, and rigid structure.

Teachers should concentrate on education, and less on the tool!

$$\int x^3 \, \mathrm{d}x = \frac{x^4}{4} - \frac{c^2}{4}$$

Sometimes we look ahead....

$$\frac{x^4}{4} - \frac{c^2}{4} = \frac{1}{4}(x^2 - c)(x^2 + c)$$

STACK and AI?

2023 annual STACK meeting
Prof. Dr. Hans-Georg Weigand's *4-step principal of inertia*

1. ignore it (1972)
2. forbid it (1974)
3. accept it reluctantly (1976)
4. make it compulsory (1978)

Mike Sharples: "Ban, Evade, Adapt, Embrace".

Preliminary thoughts

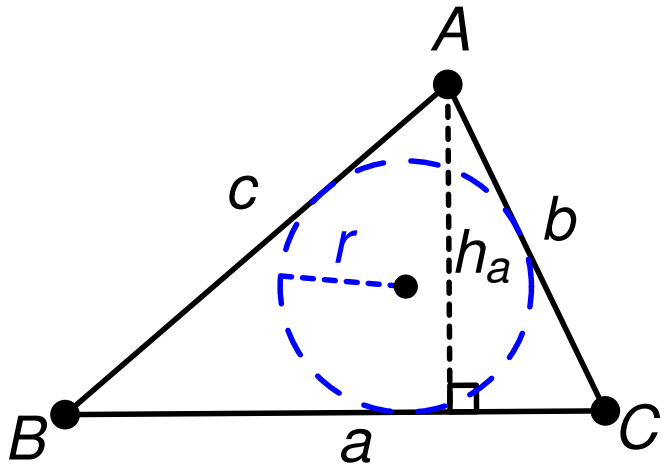
1. AI will change the subject and our relationship with it. Just as have
   - calculators, CAS,
   - computers (experiments/simlulations)
   - searchable arXiv, MathOverflow
2. We can revisit the discussions about calculators.....
3. AI is exciting!

Being a mathematician

- Solving problems
  - The classical cannon
  - Novel research
  - Engineering, physics, statistics, ...
  - Fun/recreation (including sudoku)
- Arranging and organising knowledge

Mathematics involves subjective values and emotion!

The area of a triangle



Area of triangle

$$\Delta = \frac{1}{2} a\, h_a = \frac{1}{2} bc \sin(A)$$

Define the *semi-perimeter* $s = \frac{1}{2}(a+b+c)$

$$\Delta = rs$$

There must be a formula $H(a,b,c)$

$$\Delta = \tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

Heron's formula

The cosine rule gives $-a^2 + b^2 + c^2 = 2bc\cos(A)$, so that

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}.$$

Since $\sin^2(A) + \cos^2(A) = 1$ we have

$$\begin{aligned}\sin^2(A) &= 1 - \frac{(b^2 + c^2 - a^2)^2}{4b^2c^2} \\ &= \frac{1}{4b^2c^2}(4b^2c^2 - (b^2 + c^2 - a^2)^2) \\ &= \frac{1}{4b^2c^2}(a+b+c)(-a+b+c)(a-b+c)(a+b-c).\end{aligned}$$

Since $\Delta^2 = \frac{1}{4}b^2c^2\sin^2(A)$ we have

$$\Delta = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

Insight

- Metacognition and self-awareness
- Change in understanding
  - Rapid/discontinuous: “AHA!”
  - Slow: schema building

Assume $\Delta = H(a, b, c)$. What properties must H possess?

1. H must be symmetrical in a, b and c.
2. Scale by a factor of t, area scales like t^2: $H(ta, tb, tc) = t^2 H(a, b, c)$.
3. If $c = a + b$ then the "triangle" $\rightarrow$ a straight line: $H(a, b, a + b) = 0$

$$\frac{ab}{2}\sin(C)? \quad abc? \quad (abc)^{\frac{2}{3}}? \quad a^2 + b^2 + c^2?$$

$$(a + b - c)(a - b + c)(-a + b + c)?$$

Next simplest: $\sqrt{\text{quartic}}$:

$$(a + b + c)(a + b - c)(a - b + c)(-a + b + c)$$

Consider

$$H(a, b, c) = \lambda\sqrt{(a + b + c)(a + b - c)(a + c - b)(b + c - a)} \qquad (1)$$

If $a = b = c = 1$ then $\Delta = \frac{\sqrt{3}}{4}$, giving $\lambda = \frac{1}{4}$.

Being a mathematician

Mathematics involves subjective values and emotion!

AI and surprise/emotion

Theorem A

$$1^3 + 12^3 = 1729$$

Theorem B

$$1^3 + 12^3 = 9^3 + 10^3$$

Which is more profound?

Puzzles and surprise

A man walked 5 hours, first along a level road, then up a hill, then he turned round and walked back to his starting point along the same route. He walks 4 miles per hour on the level, 3 uphill, and 6 downhill. Find the distance walked.

Knot I of “A Tangled Tale", by Lewis Carroll

- It’s surprising this problem *has* a solution!
- Mathematics often exploits special cases.

AI and organising knowledge

Algorithms are guaranteed to work for objects in a particular class.
E.g. rational expressions.

$$\int \frac{1}{x} \mathrm{d}x \notin \mathbb{Q}(x)$$

Why do we need a function from a different class, i.e. logarithms?

Closure of classes of objects lies at the heart of all mathematics
E.g. solve $x + 2 = 0$, $2x = 1$, $x^2 = 2$, $x^2 = -2$ and so on...

Only for those students interested: i.e. mathematicians.

Joel Moses (1941–)

Fundamental algorithm for integration of symbolic expressions.

(Retired....)

The overwhelming message in current calculus teaching:

if you study more calculus you can solve more problems without apparent limitations....

Mathematics curricula are very conservative

> *I was a pretty good physicist in my time. Too good — good enough to realize that all our science is just a cookery book, with an orthodox theory of cooking that nobody's allowed to question, and a list of recipes that mustn't be added to except by special permission from the head cook.*
> *Mustapha Mond*

... Brave New World

AI and STACK?

Assessment is the cornerstone of effective education.

- The subject will evolve!
  - A calculator is no longer a profession
  - Nobody teaches slide-rules
- What matters in the subject?
  Mathematics involves subjective values and emotion!
- We need assessment worth teaching to.

Conclusion

- We have sophisticated and robust tools.
- We are gaining confidence and experience in using them.
- Automating assessment has changed the way I think about the subject.
- Specific vocabulary is really important, for teaching and assessment.
- AI is an exciting and contemporary development.